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Extensions of strictly commutative Picard stacks

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EXTENSIONS OF STRICTLY COMMUTATIVE PICARD STACKS

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Abstract. Let $\mathbf{S}$ be a site. We introduce the notion of extension of strictly commutative Picard $\mathbf{S}$-stacks. Applying this notion to 1-motives, we get the notion of extension of 1-motives and we prove the following conjecture of Deligne: if $\mathcal{MR}_{\mathbb{Z}}(k)$ denotes the integral version of the neutral Tannakian category of mixed realizations over an algebraically closed field k, then the subcategory of $\mathcal{MR}_{\mathbb{Z}}(k)$ generated by 1-motives defined over k is stable under extensions.

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Introduction

In [D2] 2.4. Deligne writes: *Je conjecture que l'ensemble des motifs à coefficients entiers de la forme* $\mathrm{T}(X)$, *pour* X *un 1-motif, est stable par extensions. Si* T' *est un motif à coefficients entiers, avec* $\mathrm{T}' \otimes \mathbb{Q} \xrightarrow{\sim} \mathrm{T}(X) \otimes \mathbb{Q}$, *alors* T' *est de la forme* $\mathrm{T}(X')$ *avec* X' *isogène à* X. *La conjecture équivaut donc à ce que l'ensemble des motifs* $\mathrm{T}(X) \otimes \mathbb{Q}$, *pour* X *un 1-motif, soit stable par extension. Le mot "conjecture" est abusif en ce que l'énoncé n'a pas un sens précis. Ce qui est conjecturé est que tout système de réalizations extension de* $\mathrm{T}(X)$ *par* $\mathrm{T}(Y)$ *(*X *et* Y *deux 1-motifs), et "naturel", "provenant de la géométrie", est isomorphe à celui défini par un 1-motif* Z *extension de* X *par* Y.

The aim of this paper is to prove this conjecture.

Let $\mathbf{S}$ be a site. In the first section, to each complex of abelian sheaves over $\mathbf{S}$ concentrated in two consecutive degrees, we associate a strictly commutative Picard $\mathbf{S}$-stack as explained in [SGA4] Exposé XVIII §1.4. In section 2 we introduce the notion of extension of strictly commutative Picard $\mathbf{S}$-stacks. In Section 3 we define the notion of extension of 1-motives and we prove that an extension of 1-motives

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furnishes an extension of the corresponding strictly commutative Picard **S**-stacks. In section 4, we prove the above conjecture of Deligne, namely

Theorem 0.1. *Let k be an algebraically closed field. Let $\mathcal{MR}_{\mathbb{Z}}(k)$ be the integral version of the neutral Tannakian category over $\mathbb{Q}$ of mixed realizations (for absolute Hodge cycles) over k. Then the subcategory of $\mathcal{MR}_{\mathbb{Z}}(k)$ generated by 1-motives defined over k is stable under extensions.*

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Notation

Let E and F be two categories and let $\varphi : F \rightarrow E$ be a functor. For any S object of E, the *fibre* F_S *of* F *over* S is the sub-category of F whose arrows are the arrows m of F satisfying $\varphi(m) = id_S$. A *fibred E-category* (F, φ) is a category F endowed with a functor $\varphi : F \rightarrow E$ such that for any arrow $f : T \rightarrow S$ of E and any object y of the fibre F_S, there exists an inverse image of y via f, and such that the composite of two E-cartesian morphisms of F is again E-cartesian (see [G] Chapter I §1 for more details).

A *cartesian E-functor* $\Phi : (F, \varphi) \rightarrow (F', \varphi')$ between fibred E-categories is a functor $\Phi : F \rightarrow F'$ between the underlying categories which preserves the fibres (i.e. $\varphi' \circ \Phi = \varphi$) and which preserves E-cartesian morphisms (i.e. Φ transforms E-cartesian morphisms of F in E-cartesian morphisms of F'). An *E-morphism of cartesian E-functors* $m : \Phi \rightarrow \widetilde{\Phi}$ is a natural transformation such that $\varphi' * m$ is the identity as natural transformation of φ. We denote by

$$\mathbf{Cart}_E(F, F')$$

the category whose objects are cartesian E-functors from (F, φ) to (F', φ') and whose arrows are E-morphisms of cartesian E-functors.

Let **S**=(**S**,J) be a *site*, i.e. a category **S** endowed with a Grothendieck topology J ([G] Chapter 0 §1). In this paper the topology plays no role.

An ***S**-pre-stack* is a fibred **S**-category (F, φ) which is pre-complete, i.e. for any object T of **S** and any R element of $J(T)$ the restriction functor $\mathbf{Cart}_{\mathbf{S}}(\mathbf{S}_{|T}, F) \rightarrow \mathbf{Cart}_{\mathbf{S}}(R, F)$ is fully faithful (here $\mathbf{S}_{|T}$ is the category consisting of objects of **S** lying over T). If this restriction functor $\mathbf{Cart}_{\mathbf{S}}(\mathbf{S}_{|T}, F) \rightarrow \mathbf{Cart}_{\mathbf{S}}(R, F)$ is an equivalence of categories, the fibred **S**-category (F, φ) is complete. An ***S**-stack* is a complete fibred **S**-category (F, φ).

A *morphism of **S**-stacks* (resp. a *morphism of **S**-pre-stack*) is a cartesian **S**-functor whose source and target are **S**-stacks (resp. **S**-pre-stacks).

An ***S**-stack of groupoids* is an **S**-stack whose fibres are groupoids, i.e. categories in which every arrow is invertible.

A *2-category* $\mathcal{A} = (A, C(a, b), K_{a,b,c}, U_a)_{a,b,c \in A}$ is given by the following data:

- a set A of objects $a, b, c, \ldots$;
- for each ordered pair (a, b) of objects of A, a category $C(a, b)$;
- for each ordered triple (a, b, c) of objects A, a functor

$$K_{a,b,c} : C(b, c) \times C(a, b) \longrightarrow C(a, c),$$

called composition functor. This composition functor have to satisfy the associativity axiom which may be stated as the requirement that the following diagram be commutative

$$\begin{tikzcd} C(c,d)\times C(b,c)\times C(a,b) \arrow[r, "Id\times K_{a,b,c}"] \arrow[d, "K_{b,c,d}\times Id"'] & C(c,d)\times C(a,c) \arrow[d, "K_{a,c,d}"] \\ C(b,d)\times C(a,b) \arrow[r, "K_{a,b,d}"'] & C(a,d); \end{tikzcd}$$

- for each object a, a functor $U_a : 1 \to C(a,a)$ where 1 is the terminal category (i.e. the category with one object, one arrow), called unit functor. This unit functor have to provide a left and right identity for the composition functor, i.e. we require the commutativity of the following diagrams

$$\begin{tikzcd} C(a,a)\times C(b,a) \arrow[r, "K_{b,a,a}"] & C(b,a), \\ 1\times C(b,a) \arrow[u, "U_a\times Id"] \arrow[ur, "Id"'] & \end{tikzcd} \qquad \begin{tikzcd} C(a,b)\times C(a,a) \arrow[r, "K_{a,a,b}"] & C(a,b) \\ C(a,b)\times 1. \arrow[u, "Id\times U_a"] \arrow[ur, "Id"'] & \end{tikzcd}$$

This set of axioms for a 2-category is exactly like the set of axioms for a category in which the arrows-sets $\mathrm{Hom}(a,b)$ have been replaced by the categories $C(a,b)$. We call the categories $C(a,b)$ (with $a,b \in A$) the *categories of morphisms* of the 2-category $\mathcal{A}$: the objects of $C(a,b)$ are the *1-arrows* of $\mathcal{A}$ and the arrows of $C(a,b)$ are the *2-arrows* of $\mathcal{A}$.

Let $\mathcal{A} = (A, C(a,b), K_{a,b,c}, U_a)_{a,b,c\in A}$ and $\mathcal{A}' = (A', C(a',b'), K_{a',b',c'}, U_{a'})_{a',b',c'\in A'}$ be two 2-categories. A *2-functor* (called also a *morphism of 2-categories*)

$$(F, F_{a,b})_{a,b\in A} : \mathcal{A} \longrightarrow \mathcal{A}'$$

consists of

- an application $F : A \to A'$ between the objects of $\mathcal{A}$ and the objects of $\mathcal{A}'$,
- a family of functors $F_{a,b} : C(a,b) \to C(F(a),F(b))$ (with $a,b \in A$) which are compatible with the composition functors and with the unit functors of $\mathcal{A}$ and $\mathcal{A}'$.

Explicitly, the compatibility of the family of functors $F_{a,b}$ with the composition functors of $\mathcal{A}$ and $\mathcal{A}'$ means that the following diagram is commutative for all $a,b,c \in A$

$$\begin{tikzcd} C(b,c)\times C(a,b) \arrow[r, "K_{a,b,c}"] \arrow[d, "F_{b,c}\times F_{a,b}"'] & C(a,c) \arrow[d, "F_{a,c}"] \\ C(F(b),F(c))\times C(F(a),F(b)) \arrow[r, "K_{F(a),F(b),F(c)}"'] & C(F(a),F(c)). \end{tikzcd}$$

The compatibility of the family of functors $F_{a,b}$ with the unit functors of $\mathcal{A}$ and $\mathcal{A}'$ means that we require the following diagram be commutative for all $a \in A$

$$\begin{tikzcd}
1 \arrow[d, "U_a"'] \arrow[drr, "U_{F(a)}"] & & \\
C(a,a) \arrow[rr, "F_{a,a}"'] & & C(F(a),F(a)).
\end{tikzcd}$$

1. Picard stacks associated to a complex of abelian sheaves concentrated in two consecutive degrees

In this section we recall the dictionary between strictly commutative Picard stacks and complexes of abelian sheaves concentrated in two consecutive degrees which is explained in [SGA4] Exposé XVIII §1.4.

We first recall the definition of a strictly commutative Picard stack. Let C be a category and $\square : C \times C \to C$ a functor. Consider the two natural isomorphisms σ and τ given explicitly by the following functorial isomorphisms

$$\begin{aligned}
\sigma_{a,b,c} &: (a\square b)\square c \xrightarrow{\cong} a\square(b\square c) \\
\tau_{a,b} &: a\square b \xrightarrow{\cong} b\square a
\end{aligned}$$

for all a, b and c objects of the category C. The triplet $(\square, \sigma, \tau)$ is an *associative and strictly commutative functor* if

(1) the pentagonal axiom is satisfied, i.e. the pentagonal diagram

$$\begin{tikzcd}
 & (a\square b)\square(c\square d) \arrow[dl, "\sigma_{a,b,c\square d}"'] & \\
a\square(b\square(c\square d)) & & ((a\square b)\square c)\square d \arrow[ul, "\sigma_{a\square b,c,d}"'] \arrow[d, "\sigma_{a,b,c}\square Id"] \\
a\square((b\square c)\square d) \arrow[u, "Id\square\sigma_{b,c,d}"] & & (a\square(b\square c))\square d \arrow[ll, "\sigma_{a,b\square c,d}"]
\end{tikzcd}$$

commutes for all $a, b, c, d \in C$.

(2) $\tau_{a,a} : a\square a \to a\square a$ is the identity for all $a \in C$

(3) $\tau_{b,a} \circ \tau_{a,b} : a\square b \to b\square a \to a\square b$ is the identity for all $a, b \in C$

(4) the hexagonal axiom is satisfied, i.e. the hexagonal diagram

$$\begin{tikzcd}
 & a\square(b\square c) & \\
(a\square b)\square c \arrow[ur, "\sigma_{a,b,c}"] & & a\square(c\square b) \arrow[ul, "Id\square\tau_{c,b}"'] \\
c\square(a\square b) \arrow[u, "\tau_{c,a\square b}"] & & (a\square c)\square b \arrow[u, "\sigma_{a,c,b}"'] \\
 & (c\square a)\square b \arrow[ul, "\sigma_{c,a,b}"] \arrow[ur, "\tau_{c,a}\square Id"'] &
\end{tikzcd}$$

commutes for all $a, b, c \in C$.

Definition 1.1. A *strictly commutative Picard category* $\mathcal{P} = (\mathcal{P}, +, \sigma, \tau)$ is a non empty category $\mathcal{P}$

(1) in which every arrow is invertible;
(2) endowed with a functor, that we denote $+ : \mathcal{P} \times \mathcal{P} \to \mathcal{P},\ (a,b) \mapsto a+b$;
(3) endowed with two natural isomorphisms σ and τ (called respectively natural isomorphism of associativity and of commutativity), which are described by the functorial isomorphisms

$$\begin{aligned} \sigma_{a,b,c} \quad &: \quad (a+b)+c \xrightarrow{\cong} a+(b+c) \qquad \forall\ a,b,c \in \mathcal{P} \\ \tau_{a,b} \quad &: \quad a+b \xrightarrow{\cong} b+a \qquad \forall\ a,b \in \mathcal{P}, \end{aligned}$$

and which endow the functor $+ : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ of a structure of associative and strictly commutative functor $(+, \sigma, \tau)$;
(4) such that for any object a of $\mathcal{P}$, the functor $\mathcal{P} \to \mathcal{P},\ b \mapsto a+b$ is an equivalence of categories.

A strictly commutative Picard category $\mathcal{P}$ admits a unique, up to a unique isomorphism, neutral object that we can describe as a couple (e, φ) where e is an object of $\mathcal{P}$ and $\varphi : e+e \to e$ is an isomorphism. There is a unique natural isomorphism $\alpha_l : e+a \to a$ which makes the following diagram commutative

$$\begin{tikzcd} (e+e)+a \arrow[r, "\sigma_{e,e,a}"] \arrow[d, "\varphi+Id"'] & e+(e+a) \arrow[d, "Id+\alpha_l"] \\ e+a \arrow[r, equal] & e+a \end{tikzcd}$$

In an analogous way, there is a unique natural isomorphism $\alpha_r : a+e \to a$ which makes the following diagram commutative

$$\begin{tikzcd} (a+e)+e \arrow[r, "\sigma_{a,e,e}"] \arrow[d, "\varphi+Id"'] & a+(e+e) \arrow[d, "Id+\alpha_r"] \\ a+e \arrow[r, equal] & a+e \end{tikzcd}$$

The isomorphism φ is a special case of α_r and α_l. The natural isomorphism τ exchanges α_r and α_l, i.e. the following diagram is commutative:

$$\begin{tikzcd} a+e \arrow[r, "\alpha_r"] \arrow[d, "\tau_{a,e}"'] & a \\ e+a. \arrow[ur, "\alpha_l"'] & \end{tikzcd}$$

The group of automorphisms of the neutral object, $\mathrm{Aut}(e)$, is abelian. For any object a of $\mathcal{P}$, the functors $\mathcal{P} \to \mathcal{P},\ b \mapsto a+b$ and $\mathcal{P} \to \mathcal{P},\ b \mapsto b+a$ furnish the same isomorphism between the groups $\mathrm{Aut}(e)$ and $\mathrm{Aut}(a)$:

$$\begin{array}{rcl} \mathrm{Aut}(e) & \longrightarrow & \mathrm{Aut}(e+a) \cong \mathrm{Aut}(a+e) \cong \mathrm{Aut}(a) \\ f & \mapsto & f + id_a \cong id_a + f. \end{array} \tag{1.1}$$

Definition 1.2. A *strictly commutative Picard stack* $\mathcal{P} = (\mathcal{P}, +, \sigma, \tau)$ *over a site* $\boldsymbol{S}$ is an $\mathbf{S}$-stack of groupoids $\mathcal{P}$ endowed with

(1) a functor, that we denote $+ : \mathcal{P} \times_{\mathbf{S}} \mathcal{P} \to \mathcal{P},\ (a,b) \mapsto a+b$;

(2) two natural isomorphisms σ and τ (called natural isomorphisms of associativity and of commutativity respectively), which are described by the functorial isomorphisms

$$\sigma_{a,b,c} \quad : \quad (a+b)+c \xrightarrow{\cong} a+(b+c) \qquad \forall\ a,b,c \in \mathcal{P}, \tag{1.2}$$

$$\tau_{a,b} \quad : \quad a+b \xrightarrow{\cong} b+a \qquad \forall\ a,b \in \mathcal{P}; \tag{1.3}$$

such that for any object U of $\mathbf{S}$, $(\mathcal{P}(U),+,\sigma,\tau)$ is a strictly commutative Picard category.

Any strictly commutative Picard $\mathbf{S}$-stack admits a global neutral object e and the sheaf of automorphisms of the neutral object $\underline{\mathrm{Aut}}(e)$ is abelian. According to the isomorphism (1.1) the sheaf $\underline{\mathrm{Aut}}(e)$ is isomorphic to the sheaf of automorphisms $\underline{\mathrm{Aut}}(a)$ of any object a of $\mathcal{P}$.

Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two strictly commutative Picard $\mathbf{S}$-stacks.

Definition 1.3. An *addictive functor*

$$(F,\sum) : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$$

between strictly commutative Picard $\mathbf{S}$-stacks is a morphism of $\mathbf{S}$-stacks (i.e. a cartesian $\mathbf{S}$-functor) endowed with a natural isomorphism $\sum$ which is described by the functorial isomorphisms

$$\sum_{a,b} : F(a+b) \xrightarrow{\cong} F(a)+F(b) \qquad \forall\ a,b \in \mathcal{P}_1$$

and which is compatible with the natural isomorphisms σ and τ of $\mathcal{P}_1$ and $\mathcal{P}_2$.

Explicitly, the compatibilities of $\sum$ with τ and with σ mean respectively that the following diagrams are commutative

$$\begin{tikzcd}
F(a+b) \arrow[r, "\sum_{a,b}"] \arrow[d, "F(\tau_{a,b})"'] & F(a)+F(b) \arrow[d, "\tau_{F(a),F(b)}"] \\
F(b+a) \arrow[r, "\sum_{b,a}"'] & F(b)+F(a),
\end{tikzcd}$$

$$\begin{tikzcd}
F((a+b)+c) \arrow[r, "\sum_{a+b,c}"] \arrow[d, "F(\sigma_{a,b,c})"'] & F(a+b)+F(c) \arrow[r, "\sum_{a,b}+Id"] & (F(a)+F(b))+F(c) \arrow[d, "\sigma_{F(a),F(b),F(c)}"] \\
F(a+(b+c)) \arrow[r, "\sum_{a,b+c}"'] & F(a)+F(b+c) \arrow[r, "Id+\sum_{b,c}"'] & F(a)+(F(b)+F(c)).
\end{tikzcd}$$

Definition 1.4. A *morphism of addictive functors* $u : (F,\sum) \rightarrow (F',\sum')$ is an $\mathbf{S}$-morphism of cartesian $\mathbf{S}$-functors which is compatible with the natural isomorphisms $\sum$ and $\sum'$ of F and F' respectively, i.e. such that the following diagram is commutative for all $a, b \in \mathcal{P}_1$

$$\begin{tikzcd} F(a+b) \arrow[r, "u_{a+b}"] \arrow[d, "\sum_{a,b}"'] & F'(a+b) \arrow[d, "\sum'_{a,b}"] \\ F(a)+F(b) \arrow[r, "u_a+u_b"'] & F'(a)+F'(b). \end{tikzcd}$$

We denote by

$$\mathbf{Add}_{\mathbf{S}}(\mathcal{P}_1, \mathcal{P}_2)$$

the category whose objects are addictive functors from $\mathcal{P}_1$ to $\mathcal{P}_2$ and whose arrows are morphisms of addictive functors. The category $\mathbf{Add}_{\mathbf{S}}(\mathcal{P}_1, \mathcal{P}_2)$ is a full subcategory of the category $\mathbf{Cart}_{\mathbf{S}}(\mathcal{P}_1, \mathcal{P}_2)$:

$$\mathbf{Add}_{\mathbf{S}}(\mathcal{P}_1, \mathcal{P}_2) \subset \mathbf{Cart}_{\mathbf{S}}(\mathcal{P}_1, \mathcal{P}_2).$$

Moreover $\mathbf{Add}_{\mathbf{S}}(\mathcal{P}_1, \mathcal{P}_2)$ is a groupoid, i.e. any morphism of addictive functors is invertible in $\mathbf{Add}_{\mathbf{S}}(\mathcal{P}_1, \mathcal{P}_2)$, i.e. it is an *isomorphism of addictive functors.*

Definition 1.5. An *equivalence of strictly commutative Picard **S**-stacks* between $\mathcal{P}_1$ and $\mathcal{P}_2$ is a pair of addictive functors $(F, \sum) : \mathcal{P}_1 \to \mathcal{P}_2, (F', \sum') : \mathcal{P}_2 \to \mathcal{P}_1$ endowed with two isomorphisms of addictive functors $Id_{\mathcal{P}_1} \cong (F', \sum')\circ(F, \sum), Id_{\mathcal{P}_2} \cong (F, \sum) \circ (F', \sum')$.

Two strictly commutative Picard **S**-stacks are *equivalent as strictly commutative Picard **S**-stacks* if there exists an equivalence of strictly commutative Picard **S**-stacks between them.

To any strictly commutative Picard **S**-stack $\mathcal{P}$ we associate two sheaves:

$$\pi_0(\mathcal{P})$$

the sheaffification of the pre-sheaf which associates to each object U of **S** the group of isomorphism classes of objects of $\mathcal{P}(U)$ and

$$\pi_1(\mathcal{P})$$

the sheaf of automorphisms $\underline{\mathrm{Aut}}(e)$ of the neutral object of $\mathcal{P}$.
These two sheaves determine $\mathcal{P}$ modulo equivalences of strictly commutative Picard **S**-stacks: $\mathcal{P}$ and $\mathcal{P}'$ are equivalent as strictly commutative Picard **S**-stacks if and only if $\pi_i(\mathcal{P})$ is isomorphic to $\pi_i(\mathcal{P}')$ for $i = 0, 1$.

Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two strictly commutative Picard **S**-stacks. Let $\mathrm{HOM}(\mathcal{P}_1, \mathcal{P}_2)$ the following strictly commutative Picard **S**-stack:

- for any object U of **S**, the objects of the category $\mathrm{HOM}(\mathcal{P}_1, \mathcal{P}_2)(U)$ are addictive functors from $\mathcal{P}_{1|U}$ to $\mathcal{P}_{2|U}$ and its arrows are morphisms of addictive functors;
- the functor $+ : \mathrm{HOM}(\mathcal{P}_1, \mathcal{P}_2) \times \mathrm{HOM}(\mathcal{P}_1, \mathcal{P}_2) \to \mathrm{HOM}(\mathcal{P}_1, \mathcal{P}_2)$ is defined by the formula

  $$(F_1 + F_2)(a) = F_1(a) + F_2(a) \qquad \forall\ a \in \mathcal{P}_1$$

  and the natural isomorphism

  $$\sum : (F_1 + F_2)(a + b) \xrightarrow{\cong} (F_1 + F_2)(a) + (F_1 + F_2)(b)$$

is given by the commutative diagram

$$\begin{tikzcd}
(F_1+F_2)(a+b) \arrow[r, "\sum"] \arrow[d, equal] & (F_1+F_2)(a)+(F_1+F_2)(b) \arrow[r, equal] & F_1(a)+F_2(a)+F_1(b)+F_2(b) \\
F_1(a+b)+F_2(a+b) \arrow[rr, "\sum_{F_1}+\sum_{F_2}"'] & & F_1(a)+F_1(b)+F_2(a)+F_2(b). \arrow[u, "Id+\tau_{F_1(b),F_2(a)}+Id"']
\end{tikzcd}$$

- the natural isomorphisms of associativity σ and of commutativity τ of $\mathrm{HOM}(\mathcal{P}_1, \mathcal{P}_2)$ are defined via the analogous natural isomorphisms of $\mathcal{P}_2$.

Definition 1.6. A *strictly commutative Picard pre-stack* $\mathcal{P} = (\mathcal{P}, +, \sigma, \tau)$ *over a site* $\boldsymbol{S}$ is an $\mathbf{S}$-pre-stack of groupoids $\mathcal{P}$ endowed with a functor $+ : \mathcal{P} \times_{\mathbf{S}} \mathcal{P} \to \mathcal{P}$ and two natural isomorphisms σ (1.2) and τ (1.3), such that for any object U of $\mathbf{S}$, $(\mathcal{P}(U), +, \sigma, \tau)$ is a strictly commutative Picard category.

We define addictive functors $(F, \sum) : \mathcal{P}_1 \to \mathcal{P}_2$ between strictly commutative Picard $\mathbf{S}$-pre-stacks as in Definition 1.3 and we denote by $\mathrm{HOM}(\mathcal{P}_1, \mathcal{P}_2)$ the strictly commutative Picard $\mathbf{S}$-pre-stack that they form. According to [SGA4] Exposé XVIII 1.4.10, if $\mathcal{P}$ is a strictly commutative Picard $\mathbf{S}$-pre-stack, there exists modulo a unique equivalence one and only one couple $(a\mathcal{P}, j)$ where $a\mathcal{P}$ is a strictly commutative Picard $\mathbf{S}$-stack and $j : \mathcal{P} \to a\mathcal{P}$ is an addictive functor. This couple $(a\mathcal{P}, j)$ is called *the strictly commutative Picard* $\boldsymbol{S}$*-stack generated by* $\mathcal{P}$. For any strictly commutative Picard $\mathbf{S}$-stack $\mathcal{P}_1$, we have the following equivalence

$$\mathrm{HOM}(a\mathcal{P}, \mathcal{P}_1) \xrightarrow{\cong} \mathrm{HOM}(\mathcal{P}, \mathcal{P}_1). \tag{1.4}$$

Denote by $\mathcal{K}(\mathbf{S})$ the category of complexes of abelian sheaves over the site $\mathbf{S}$: all complexes that we consider in this paper are cochain complexes. Let $\mathcal{K}^{[-1,0]}(\mathbf{S})$ the subcategory of $\mathcal{K}(\mathbf{S})$ of complexes $K = (K^i)_i$ such that $K^i = 0$ for $i \neq -1$ or 0. As in [SGA4] Exposé XVIII 1.4.11 to each complex

$$K = [K^{-1} \stackrel{d}{\longrightarrow} K^0]$$

of $\mathcal{K}^{[-1,0]}(\mathbf{S})$ we associate a strictly commutative Picard $\mathbf{S}$-pre-stack $pst(K)$ as followed:

(1) for any object U of $\mathbf{S}$, the objects of $pst(K)(U)$ are the elements of $K^0(U)$;
(2) for any object U of $\mathbf{S}$, if x and y are two objects of $pst(K)(U)$ (i.e. $x, y \in K^0(U)$), an arrow of $pst(K)(U)$ from x to y is an element f of $K^{-1}(U)$ such that $df = y - x$;
(3) the composition of arrows in $pst(K)(U)$ is the additive law of $K^{-1}(U)$;
(4) the functor $+ : pst(K) \times pst(K) \to pst(K)$ is given by the additive law of K^{-1} and K^0;
(5) the natural isomorphisms of associativity σ and of commutativity τ are furnished by the neutral element of K^{-1}.

The strictly commutative Picard $\mathbf{S}$-stack associated to the complex K

$$st(K)$$

is the strictly commutative Picard $\mathbf{S}$-stack generated by $pst(K)$. The strictly commutative Picard $\mathbf{S}$-pre-stack $pst([K^{-1} \stackrel{d}{\to} K^0])$ can be described as the $\mathbf{S}$-pre-stack of trivial K^{-1}-torsors such that the K^0-torsors they define by extension of the structural group via d are trivialized (see [G] Chapter III §1.3). Therefore $st([K^{-1} \xrightarrow{d} K^0])$ can be identified with the **S**-stack of K^{-1}-torsors such that the K^0-torsors they define by extension of the structural group via d are trivialized.

Lemma 1.7. *If $K = [K^{-1} \xrightarrow{d} K^0]$ is a complex of $\mathcal{K}^{[-1,0]}(\boldsymbol{S})$, then*

$$\begin{aligned} \pi_0(st(K)) &= \mathrm{H}^0(K) \\ \pi_1(st(K)) &= \mathrm{H}^{-1}(K). \end{aligned}$$

Proof. By definition, an element f of $K^{-1}(U)$ (with U an object of **S**) is an automorphism of an object x of $K^0(U)$ if

$$df = x - x.$$

This equality means that the sheaf of automorphism $\underline{\mathrm{Aut}}(x)$ of any object x of $st(K)$ is the kernel of d, i.e. $\pi_1(st(K)) = \mathrm{H}^{-1}(K)$. For any object U of **S**, all the arrows of the category $st(K)(U)$ are invertible (the inverse of $f \in K^{-1}(U)$ is $-f$). Therefore the group of isomorphism classes of objects of $st(K)(U)$ is the group $K^0(U)/dK^{-1}(U)$, which implies that $\pi_0(st(K)) = \mathrm{H}^0(K)$. □

Let $K = [K^{-1} \xrightarrow{d} K^0]$ and $L = [L^{-1} \xrightarrow{d} L^0]$ be two complexes of $\mathcal{K}^{[-1,0]}(\mathbf{S})$. A morphism of complexes $f = (f^{-1}, f^0) : K \to L$ induces an addictive functor $pst(f) : pst(K) \to pst(L)$ between the strictly commutative Picard **S**-pre-stacks associated to the two complexes K and L, and so an addictive functor

$$st(f) : st(K) \longrightarrow st(L)$$

between strictly commutative Picard **S**-stacks. As a consequence of Lemma 1.7 we have the following

Lemma 1.8. *The addictive functor $st(f)$ is an equivalence of strictly commutative Picard $\boldsymbol{S}$-stacks if and only if f is a quasi-isomorphism.*

Proposition 1.9. *If $f, g : K \to L$ are two morphisms of complexes of $\mathcal{K}^{[-1,0]}(\boldsymbol{S})$, then*

$$\mathrm{Hom}_{\mathbf{Add}_{\boldsymbol{S}}}(st(f), st(g)) \xrightarrow{\cong} \Big\{ \text{homotopies } H : K \to L \mid g - f = dH + Hd \Big\}.$$

Proof. A morphism of addictive functors $h : pst(f) \to pst(g)$ is a morphism of abelian sheaves $h : K^0 \to L^{-1}$ such that for any object U of **S**,

- if x is an object of $pst(K)(U)$ (i.e. $x \in K^0(U)$), $h(x) \in L^{-1}(U)$ is an arrow of $pst(L)(U)$ from $f(x)$ to $g(x)$, i.e.

$$g(x) - f(x) = dh(x). \tag{1.5}$$

- if $u \in K^{-1}(U)$ is an arrow of $pst(K)(U)$ from x to y (i.e. $y - x = du$), the following diagram is commutative

$$\begin{tikzcd} f(x) \arrow[r, "h(x)"] \arrow[d, "f(u)"'] & g(x) \arrow[d, "g(u)"] \\ f(y) \arrow[r, "h(y)"'] & g(y), \end{tikzcd}$$

which means that

$$h(y) + f(u) = g(u) + h(x) \tag{1.6}$$

since the composition of arrows in $pst(L)(U)$ is the additive law of $L^{-1}(U)$.
The fact that $h : pst(f) \to pst(g)$ is a morphism of addictive functors implies that for any object U of $\mathbf{S}$ and for any objects x, y of $pst(K)(U)$

$$h(x+y) = h(x) + h(y).$$

Using this last equality, we can rewrite the equality (1.6) as followed

$$g(u) - f(u) = h(y-x) = h(du). \tag{1.7}$$

The equalities (1.5) and (1.7) means that $g - f = dh + hd$, i.e. $h : pst(f) \to pst(g)$ is an homotopy between f and g. According to (1.4) we have that

$$\mathrm{Hom}_{\mathbf{Add_S}}(st(f), st(g)) \stackrel{\cong}{\longrightarrow} \mathrm{Hom}_{\mathbf{Add_S}}(pst(f), pst(g))$$

and so we can conclude. $\square$

Denote by $\mathbf{Picard}(\mathbf{S})$ the category whose objects are strictly commutative Picard $\mathbf{S}$-stacks and whose arrows are isomorphism classes of addictive functors. Let $\mathcal{D}(\mathbf{S})$ be the derived category of the category of abelian sheaves over $\mathbf{S}$, and let $\mathcal{D}^{[-1,0]}(\mathbf{S})$ be the subcategory of $\mathcal{D}(\mathbf{S})$ consisting of complexes K such that $\mathrm{H}^i(K) = 0$ for $i \neq -1$ or 0. In [SGA4] Exposé XVIII Lemma 1.4.13 Deligne proved that, for any strictly commutative Picard $\mathbf{S}$-stack $\mathcal{P}$ there exists a complex of $\mathcal{K}^{[-1,0]}(\mathbf{S})$ such that $\mathcal{P} = st(K)$. Therefore using Lemma 1.8 and Proposition 1.9, we obtain the following

Proposition 1.10. *The functor*

$$\begin{array}{rcl} st : \mathcal{D}^{[-1,0]}(\boldsymbol{S}) & \longrightarrow & \mathbf{Picard}(\boldsymbol{S}) \\ K & \mapsto & st(K) \\ K \stackrel{f}{\to} L & \mapsto & st(K) \stackrel{st(f)}{\to} st(L) \end{array} \tag{1.8}$$

is an equivalence of categories.

We denote by [ ] the inverse equivalence of st. If $\mathcal{P}$ is a strictly commutative Picard $\mathbf{S}$-stack and $[\mathcal{P}] \in \mathcal{D}^{[-1,0]}(\mathbf{S})$ is the corresponding complex, by Lemma 1.7

$$\begin{array}{rcl} \pi_0(\mathcal{P}) & = & \mathrm{H}^0([\mathcal{P}]), \\ \pi_1(\mathcal{P}) & = & \mathrm{H}^{-1}([\mathcal{P}]). \end{array}$$

Using [SGA4] Exposé XVIII 1.4.16, we have

Corollary 1.11. *Via the functor st, there exists a 2-functor between*

- **(a):** *the 2-category of strictly commutative Picard $\boldsymbol{S}$-stacks whose objects are strictly commutative Picard $\boldsymbol{S}$-stacks and whose categories of morphisms are the categories $\mathbf{Add}_{\boldsymbol{S}}(\mathcal{P}_1, \mathcal{P}_2)$ (i.e. the 1-arrows are addictive functors between strictly commutative Picard $\boldsymbol{S}$-stacks and the 2-arrows are morphisms of addictive functors)*
- **(b):** *the 2-category whose objects and 1-arrows are the objects and the arrows of the category $\mathcal{K}^{[-1,0]}(\boldsymbol{S})$ and whose 2-arrows are the homotopies between 1-arrows (i.e. H such that $g - f = dH + Hd$ with $f, g : K \to L$ 1-arrows).*

Moreover, this 2-functor is an equivalence of 2-categories if in (b) we restrict to the full sub-category of $\mathcal{K}^{[-1,0]}(\boldsymbol{S})$ consisting of the complexes K with K^{-1} injective.

2. Extensions of strictly commutative Picard stacks

Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two strictly commutative Picard stacks over a site $\mathbf{S}$. Consider an addictive functor

$$(F, \sum) : \mathcal{P}_1 \to \mathcal{P}_2$$

from $\mathcal{P}_1$ to $\mathcal{P}_2$.

Definition 2.1. The *kernel of* F is the strictly commutative Picard $\mathbf{S}$-stack $\ker(F) = (\ker(F), +, \sigma, \tau)$ where

- for any object U of $\mathbf{S}$, the category $\ker(F)(U)$ is the full subcategory of $\mathcal{P}_1(U)$ consisting of the objects p_1 endowed with an isomorphism between $F(p_1)$ and the neutral object e of $\mathcal{P}_2$: $F(p_1) \cong e$;
- the functor $+ : \ker(F) \times \ker(F) \to \ker(F)$ is the restriction to $\ker(F)$ of the functor $+ : \mathcal{P}_1 \times \mathcal{P}_1 \to \mathcal{P}_1$ of $\mathcal{P}_1$;
- the natural isomorphisms of associativity σ and of commutativity τ are the restrictions to $\ker(F)$ of the analogous natural isomorphisms of $\mathcal{P}_1$.

Remark that the functor $+ : \ker(F) \times \ker(F) \to \ker(F)$ is well defined. In fact if p_1 and p_1' are two objects of $\ker(F)$ then $p_1 + p_1'$ is an object of $\ker(F)$:

$$F(p_1 + p_1') \cong F(p_1) + F(p_1') \cong e + e \cong e.$$

Definition 2.2. The *cokernel of* F is the strictly commutative Picard $\mathbf{S}$-stack $\mathrm{coker}(F) = (\mathrm{coker}(F), +, \sigma, \tau)$ generated by the following strictly commutative Picard $\mathbf{S}$-pre-stack $\mathrm{coker}'(F) = (\mathrm{coker}'(F), +, \sigma, \tau)$ where

- for any object U of $\mathbf{S}$, the objects of $\mathrm{coker}'(F)(U)$ are the objects of $\mathcal{P}_2(U)$;
- for any object U of $\mathbf{S}$, if p_2' and p_2'' are two objects of $\mathrm{coker}'(F)(U)$ (i.e. objects of $\mathcal{P}_2(U)$), an arrow of $\mathrm{coker}'(F)(U)$ from p_2' to p_2'' is an isomorphism class of pairs (p_1, α) with p_1 an object of $\mathcal{P}_1(U)$ and $\alpha : p_2' + F(p_1) \to p_2''$ an arrow of $\mathcal{P}_2(U)$;
- the functor $+ : \mathrm{coker}'(F) \times \mathrm{coker}'(F) \to \mathrm{coker}'(F)$ is the restriction to $\mathrm{coker}'(F)$ of the functor $+ : \mathcal{P}_2 \times \mathcal{P}_2 \to \mathcal{P}_2$ of $\mathcal{P}_2$;
- the natural isomorphisms of associativity σ and of commutativity τ are the restrictions to $\mathrm{coker}(F)$ of the analogous natural isomorphisms of $\mathcal{P}_2$.

Denote respectively by $[\mathcal{P}_1] = [K^{-1} \stackrel{d^K}{\to} K^0]$ and $[\mathcal{P}_2] = [L^{-1} \stackrel{d^L}{\to} L^0]$ the complexes of $\mathcal{D}^{[-1,0]}(\mathbf{S})$ corresponding to the strictly commutative Picard $\mathbf{S}$-stacks $\mathcal{P}_1$ and $\mathcal{P}_2$ via the equivalence of categories *st* (1.8). Let

$$f = (f^{-1}, f^0) = [F] : [\mathcal{P}_1] \longrightarrow [\mathcal{P}_2]$$

be the morphism of complexes corresponding to the addictive functor $F:\mathcal{P}_1\to\mathcal{P}_2$. Consider the following bi-complex:

$$\begin{array}{ccccccccc} & & \overbrace{\phantom{L^{-1}}}^{-1*} & & \overbrace{\phantom{L^{0}}}^{0*} & & & & \\ & & 0 & & 0 & & & & \\ & & \uparrow & & \uparrow & & & & \\ 0 & \to & L^{-1} & \stackrel{d^L}{\to} & L^0 & \to & 0 & \} & *1 \\ & & {\scriptstyle f^{-1}}\uparrow & & \uparrow{\scriptstyle f^0} & & & & \\ 0 & \to & K^{-1} & \stackrel{d^K}{\to} & K^0 & \to & 0 & \} & *0 \\ & & \uparrow & & \uparrow & & & & \\ & & 0 & & 0 & & & & \end{array} \tag{2.1}$$

where L^{-1},L^0,K^{-1} and K^0 are respectively in bi-degrees (-1,1), (0,1),(-10) and (0,0). The total complex associated to this bi-complex is the complex $\mathbb{F}=(\mathbb{F}^i)_i$ such that $\mathbb{F}^{-1}=K^{-1},\mathbb{F}^0=L^{-1}+K^0,\mathbb{F}^1=L^0$ and $\mathbb{F}^i=0$ for $i\neq -1,0,1$:

$$\mathbb{F}: \qquad 0\longrightarrow K^{-1} \stackrel{(f^{-1},d^K)}{\longrightarrow} L^{-1}+K^0 \stackrel{(d^L,f^0)}{\longrightarrow} L^0\longrightarrow 0.$$

The complexes of $\mathcal{D}^{[-1,0]}(\mathbf{S})$ corresponding to the strictly commutative Picard $\mathbf{S}$-stacks kernel of F and cokernel of F via the equivalence of categories (1.8) are respectively

$$[\ker(F)] = \tau_{\leq 0}\mathbb{F}=\big[K^{-1} \stackrel{\overline{(f^{-1},d^K)}}{\longrightarrow} \ker(d^L,f^0)\big] \tag{2.2}$$

$$[\operatorname{coker}(F)] = (\tau_{\geq 0}\mathbb{F})[1]=\big[\operatorname{coker}(f^{-1},d^K) \stackrel{\overline{(d^L,f^0)}}{\longrightarrow} L^0\big] \tag{2.3}$$

where τ denotes the good truncation and $\overline{(f^{-1},d^K)}:K^{-1}\to\ker(d^L,f^0)$ and $\overline{(d^L,f^0)}:\operatorname{coker}(f^{-1},d^K)\to L^0$ are the morphisms of sheaves induced respectively by $(f^{-1},d^K):K^{-1}\to L^{-1}+K^0$ and $(d^L,f^0):L^{-1}+K^0\to L^0$.

Clearly the morphism of complexes $f=(f^{-1},f^0)=[F]:[\mathcal{P}_1]\to[\mathcal{P}_2]$ induces morphisms between the sheaves π_0 and π_1 associated to the strictly commutative Picard $\mathbf{S}$-stacks $\mathcal{P}_1$ and $\mathcal{P}_2$:

$$\begin{aligned} \pi_0(F):\pi_0(\mathcal{P}_1)=K^0/d^KK^{-1} &\longrightarrow \pi_0(\mathcal{P}_2)=L^0/d^LL^{-1},\\ \pi_1(F):\pi_1(\mathcal{P}_1)=\ker(d^K) &\longrightarrow \pi_1(\mathcal{P}_2)=\ker(d^L). \end{aligned}$$

According to the first equality (2.2), we have that

$$\begin{aligned} \pi_0(\ker(F)) &= \ker(d^L,f^0)/\mathrm{im}\overline{(f^{-1},d^K)},\\ \pi_1(\ker(F)) &= \ker\overline{(f^{-1},d^K)}. \end{aligned}$$

Therefore the strictly commutative Picard $\mathbf{S}$-stacks $\ker(F)$ and $\mathcal{P}_1$ have the same π_1, i.e. the same sheaf of automorphisms of the neutral object,

$$\pi_1(\ker(F))=\pi_1(\mathcal{P}_1) \tag{2.4}$$

and the sheaf $\pi_0(\ker(F))$ is the kernel of the morphism $\pi_0(F):\pi_0(\mathcal{P}_1)\to\pi_0(\mathcal{P}_2)$, i.e. the following sequence of sheaves is exact

$$0\longrightarrow \pi_0(\ker(F)) \longrightarrow \pi_0(\mathcal{P}_1) \stackrel{\pi_0(F)}{\longrightarrow} \pi_0(\mathcal{P}_2). \tag{2.5}$$

The second equality (2.3) implies that

$$\begin{aligned}\pi_0(\mathrm{coker}(F)) &= L^0/\mathrm{im}\overline{(d^L, f^0)},\\ \pi_1(\mathrm{coker}(F)) &= \ker\overline{(d^L, f^0)}.\end{aligned}$$

Hence the strictly commutative Picard $\mathbf{S}$-stacks $\mathrm{coker}(F)$ and $\mathcal{P}_2$ have the same π_0

$$\pi_0(\mathrm{coker}(F)) = \pi_0(\mathcal{P}_2) \tag{2.6}$$

and the sheaf $\pi_1(\mathrm{coker}(F))$ is the cokernel of the morphism $\pi_1(F) : \pi_1(\mathcal{P}_1) \to \pi_1(\mathcal{P}_2)$, i.e. the following sequence of sheaves is exact

$$\pi_1(\mathcal{P}_1) \xrightarrow{\pi_1(F)} \pi_1(\mathcal{P}_2) \longrightarrow \pi_1(\mathrm{coker}(F)) \longrightarrow 0. \tag{2.7}$$

Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two strictly commutative Picard $\mathbf{S}$-stacks.

Definition 2.3. An *extension* $\mathcal{P} = (\mathcal{P}, I : \mathcal{P}_2 \to \mathcal{P}, J : \mathcal{P} \to \mathcal{P}_1)$ *of* $\mathcal{P}_1$ *by* $\mathcal{P}_2$

$$\mathcal{P}_2 \xrightarrow{I} \mathcal{P} \xrightarrow{J} \mathcal{P}_1$$

consists of a strictly commutative Picard $\mathbf{S}$-stack $\mathcal{P}$ and two addictive functors $I : \mathcal{P}_2 \to \mathcal{P}$ and $J : \mathcal{P} \to \mathcal{P}_1$ such that

- there exists an isomorphism of addictive functors between the composite $J \circ I$ and the trivial addictive functor: $J \circ I \cong 0$,
- the following equivalent conditions are satisfied:
  - **(a):** $\pi_0(J) : \pi_0(\mathcal{P}) \to \pi_0(\mathcal{P}_1)$ is surjective and I induces an equivalence of strictly commutative Picard $\mathbf{S}$-stacks between $\mathcal{P}_2$ and $\ker(J)$;
  - **(b):** $\pi_1(I) : \pi_1(\mathcal{P}_2) \to \pi_1(\mathcal{P})$ is injective and J induces an equivalence of strictly commutative Picard $\mathbf{S}$-stacks between $\mathrm{coker}(I)$ and $\mathcal{P}_1$.

Lemma 2.4. *$\mathcal{P} = (\mathcal{P}, I : \mathcal{P}_2 \to \mathcal{P}, J : \mathcal{P} \to \mathcal{P}_1)$ is an extension of $\mathcal{P}_1$ by $\mathcal{P}_2$ if and only if the composite $J \circ I$ is isomorphic to the trivial addictive functor and the following equivalent conditions are satisfied:*

- *I induces an isomorphism of sheaves between $\pi_1(\mathcal{P}_2)$ and $\pi_1(\mathcal{P})$, and I and J induce the following exact sequence of sheaves*

$$0 \longrightarrow \pi_0(\mathcal{P}_2) \xrightarrow{\pi_0(I)} \pi_0(\mathcal{P}) \xrightarrow{\pi_0(J)} \pi_0(\mathcal{P}_1) \longrightarrow 0;$$

- *J induces an isomorphism of sheaves between $\pi_0(\mathcal{P})$ and $\pi_0(\mathcal{P}_1)$, and I and J induce the following exact sequence of sheaves*

$$0 \longrightarrow \pi_1(\mathcal{P}_2) \xrightarrow{\pi_1(I)} \pi_1(\mathcal{P}) \xrightarrow{\pi_1(J)} \pi_1(\mathcal{P}_1) \longrightarrow 0.$$

Proof. By duality we just have to prove one of the two assertions. The addictive functor I induces an equivalence of strictly commutative Picard $\mathbf{S}$-stacks between $\mathcal{P}_2$ and $\ker(J)$ if and only if the morphisms of sheaves

$$\begin{aligned}\pi_0(I) : \pi_0(\mathcal{P}_2) &\longrightarrow \pi_0(\ker(J)),\\ \pi_1(I) : \pi_1(\mathcal{P}_2) &\longrightarrow \pi_1(\ker(J))\end{aligned}$$

are isomorphisms. Therefore the first assertion of this Lemma is a consequence of (2.4) and (2.5). $\square$

The extensions of $\mathcal{P}_1$ by $\mathcal{P}_2$ form a 2-category where the equivalence classes of objects, the isomorphism classes of addictive functors from an object to itself and the automorphisms of an addictive functor between objects are respectively the $\mathrm{H}^1, \mathrm{H}^0$ and H^{-1} of the complex $\mathbb{R}\mathrm{Hom}([\mathcal{P}_1], [\mathcal{P}_2])$ (see [B1]).

3. Extensions of 1-motives

Let S be a scheme.

A *1-motive* $M = (X, A, T, G, u)$ over S consists of

- an S-group scheme X which is locally for the étale topology a constant group scheme defined by a finitely generated free $\mathbb{Z}$-module,
- an extension G of an abelian S-scheme A by an S-torus T,
- a morphism $u : X \to G$ of S-group schemes.

A 1-motive $M = (X, A, T, G, u)$ can be viewed also as a complex $[X \stackrel{u}{\to} G]$ of commutative S-group schemes with X concentrated in degree -1 and G concentrated in degree 0. A morphism of 1-motives is a morphism of complexes of commutative S-group schemes. Denote by $1-\mathrm{Mot}(S)$ the category of 1-motives over S. It is an additive category but *it isn't an abelian category.*

Let $S = \mathrm{Spec}\,(k)$ be the spectrum of an algebraically closed field k. To the category $1-\mathrm{Mot}(k)$, we can associate the $\mathbb{Q}$-linear category $1-\mathrm{Isomot}(k)$ of 1-isomotifs in the following way: the category $1-\mathrm{Isomot}(k)$ has the same objects as the category $1-\mathrm{Mot}(k)$, but the sets of morphisms of $1-\mathrm{Isomot}(k)$ are the sets of morphisms of $1-\mathrm{Mot}(k)$ tensored with $\mathbb{Q}$, i.e. $\mathrm{Hom}_{1-\mathrm{Isomot}(k)}(-,-) = \mathrm{Hom}_{1-\mathrm{Mot}(k)}(-,-) \otimes_{\mathbb{Z}} \mathbb{Q}$. The objects of $1-\mathrm{Isomot}(k)$ are called 1-isomotives and the morphisms of $1-\mathrm{Mot}(k)$ which become isomorphisms in $1-\mathrm{Isomot}(k)$ are the isogenies, i.e. the morphisms of complexes $(f_{-1}, f_0) : [X \to G] \to [X' \to G']$ such that $f_{-1} : X \to X'$ is injective with finite cokernel and $f_0 : G \to G'$ is surjective with finite kernel. *The category* $1-\mathrm{Isomot}(k)$ *is an abelian category.*

If S is the spectrum of the field $\mathbb{C}$ of complex numbers, the category of 1-motives over $\mathbb{C}$ is equivalent to the category of $\mathbb{Q}$-mixed Hodge structures H, endowed with a torsion-free $\mathbb{Z}$-lattice, of type $(0;0); (-1;0); (0;-1); (-1;-1)$, and with the quotient $\mathrm{Gr}^W_{-1}(\mathrm{H})$ polarizable (see [D1] (10.1.3)). In particular, *the category* $1-\mathrm{Mot}(\mathbb{C})$ *is an abelian category.*

In [B2] the author is proving that if S is a scheme of finite type over $\mathbb{C}$, the category of 1-motives over S is equivalent to an adequate subcategory of the category of variations of mixed Hodge structures over the analytic space S^{an}. Also in this case *the category* $1-\mathrm{Mot}(S)$ *is an abelian category.*

The results of this section are true for any base scheme S such that the category $1-\mathrm{Mot}(S)$ of 1-motives over S is abelian. Let $M_1 = [X_1 \stackrel{u_1}{\to} G_1]$ and $M_2 = [X_2 \stackrel{u_2}{\to} G_2]$ be two 1-motives defined over such a base scheme S.

Definition 3.1. An extension of M_1 by M_2 consists of a 1-motive $M = [X \stackrel{u}{\to} G]$ defined over S and two morphisms of 1-motives $(i_{-1}, i_0) : M_2 \to M$ and $(j_{-1}, j_0) : M \to M_1$

(3.1)
$$\begin{tikzcd} X_2 \arrow[r, hook, "i_{-1}"] \arrow[d, "u_2"'] & X \arrow[r, two heads, "j_{-1}"] \arrow[d, "u"] & X_1 \arrow[d, "u_1"] \\ G_2 \arrow[r, hook, "i_0"] & G \arrow[r, two heads, "j_0"] & G_1 \end{tikzcd}$$

such that $j_{-1} \circ i_{-1} = 0, j_0 \circ i_0 = 0$, i_{-1} and i_0 are injective, j_{-1} and j_0 are surjective, and u induces an isomorphism between the quotients $\ker(j_{-1})/\mathrm{im}(i_{-1})$ and $\ker(j_0)/\mathrm{im}(i_0)$ (here $\mathrm{im}(i_{-1})$ and $\mathrm{im}(i_0)$ are the images of the injections i_{-1} and i_0 respectively).

Remark that according to this definition, an extension of $M_1 = [0 \stackrel{0}{\to} G_1]$ by $M_2 = [0 \stackrel{0}{\to} G_2]$ is an exact sequence

$$0 \longrightarrow G_2 \longrightarrow G \longrightarrow G_1 \longrightarrow 0.$$

Similarly, an extension of $M_1 = [X_1 \stackrel{0}{\to} 0]$ by $M_2 = [X_2 \stackrel{0}{\to} 0]$ is an exact sequence

$$0 \longrightarrow X_2 \longrightarrow X \longrightarrow X_1 \longrightarrow 0.$$

EXAMPLE: Let $M_1 = [0 \to \mathbb{G}_m]$ and $M_2 = [\mathbb{Z} \to 0]$ be two 1-motives defined over S. Consider the following extension $M = [\mathbb{Z} \stackrel{-1}{\to} \mathbb{G}_m]$ of M_1 by M_2:

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "2"] \arrow[d] & \mathbb{Z} \arrow[r, "0"] \arrow[d, "-1"'] & 0 \arrow[d] \\ 0 \arrow[r] & \mathbb{G}_m \arrow[r, "x^2"] & \mathbb{G}_m \end{tikzcd} \tag{3.2}$$

In particular the cohomology groups of the first row are isomorphic to the cohomology groups of the second row:

$$-1 : \mathbb{Z}/2\mathbb{Z} \stackrel{\cong}{\longrightarrow} \ker(x^2). \tag{3.3}$$

If we look at 1-motives as complexes of abelian sheaves concentrated in degrees -1 and 0, using the equivalence of categories (1.8) the morphisms of 1-motives $(2,0) : M_2 \to M$ and $(0,x^2) : M \to M_1$ underlying the diagram (3.2) furnish two morphisms of strictly commutative Picard $\mathbf{S}$-stacks $I : st(M_2) \to st(M)$ and $J : st(M) \to st(M_1)$. Now remark that the isomorphism (3.3) implies the exact sequence

$$0 \longrightarrow \mathbb{Z} \stackrel{2}{\longrightarrow} \mathbb{Z} \stackrel{-1}{\longrightarrow} \ker(x^2) \longrightarrow 0.$$

Therefore the multiplication by 2 induces a quasi-isomorphism between the complexes $[\mathbb{Z} \to 0]$ and $[\mathbb{Z} \stackrel{-1}{\to} \ker(x^2)]$. But according to (2.2) the complex $[\mathbb{Z} \stackrel{-1}{\to} \ker(x^2)]$ is $[\ker(J)]$ and so in the derived category $\mathcal{D}^{[-1,0]}(\mathbf{S})$ we have the equality $[\mathbb{Z} \to 0] = [\ker(J)]$, i.e. I induces an equivalence of strictly commutative Picard $\mathbf{S}$-stacks between $st(M_2)$ and $\ker(J)$. Moreover remark that since $x^2 : \mathbb{G}_m \to \mathbb{G}_m$ is surjective, also the morphism $\pi_0(J) : \pi_0(st(M)) \to \pi_0(st(M_1))$ is surjective, i.e. J is surjective on π_0. We can therefore conclude that the strictly commutative Picard $\mathbf{S}$-stack $st(M)$ is an extension of the strictly commutative Picard $\mathbf{S}$-stack $st(M_1)$ by the strictly commutative Picard $\mathbf{S}$-stack $st(M_2)$.

Now we prove that what we have observed in the above example is true in general:

Proposition 3.2. *Let $M_1 = [X_1 \stackrel{u_1}{\to} G_1]$ and $M_2 = [X_2 \stackrel{u_2}{\to} G_2]$ be two 1-motives defined over S. Let $M = [X \stackrel{u}{\to} G]$ be an extension of M_1 by M_2. Via the equivalence of categories (1.8), the strictly commutative Picard $\mathbf{S}$-stack $st(M)$ is an extension of the strictly commutative Picard $\mathbf{S}$-stack $st(M_1)$ by the strictly commutative Picard $\mathbf{S}$-stack $st(M_2)$.*

Proof. Denote by $(i_{-1}, i_0) : M_2 \to M$ and $(j_{-1}, j_0) : M \to M_1$ the morphisms of 1-motives underlying the extension M of M_1 by M_2. Using the equivalence of categories (1.8), the morphisms of 1-motives (i_{-1}, i_0) and (j_{-1}, j_0) furnish two morphisms of strictly commutative Picard $\mathbf{S}$-stacks:

$$I : st(M_2) \longrightarrow st(M) \qquad \text{and} \qquad J : st(M) \longrightarrow st(M_1).$$

First observe that the conditions $j_{-1} \circ i_{-1} = 0$ and $j_0 \circ i_0 = 0$ on the morphisms of 1-motives (i_{-1}, i_0) and (j_{-1}, j_0) imply that $J \circ I = 0$. Remark also that since $j_0 : G \to G_1$ is surjective, also the morphism $\pi_0(J) : G/u(X) \to G_1/u_1(X_1)$ is surjective, i.e. J is surjective on π_0.
By Lemma 2.4 it remains to prove that via the morphism I, $\pi_1(st(M_2))$ and $\pi_1(st(M))$ are isomorphic and $\pi_0(st(M_2))$ is the kernel of the surjection $\pi_0(J) : \pi_0(st(M)) \to \pi_0(st(M_1))$. In order to show that $\pi_1(st(M_2))$ and $\pi_1(st(M))$ are isomorphic, by (2.4) it is enough to prove that $\pi_1(st(M_2))$ and $\pi_1(\ker(J))$ are isomorphic. According to (2.2) the kernel of J is the complex

$$[X \stackrel{\overline{(j_{-1},u)}}{\longrightarrow} \ker(u_1, j_0)]$$

$\overline{(j_{-1}, u)} : X \to \ker(u_1, j_0)$ is the morphism induced by $(j_{-1}, u) : X \to X_1 + G$. Hence we have to prove that (i_{-1}, i_0) induces the following isomorphism:

$$\ker(u_2) \cong \ker \overline{(j_{-1}, u)}$$

Because of the commutativity of the first square of diagram (3.1) we have that $i_{-1}(\ker(u_2))$ is contained in $\ker(u)$. Since $j_{-1} \circ i_{-1} = 0$ we have also that $i_{-1}(\ker(u_2))$ is contained in $\ker(j_{-1})$. Therefore we get the inclusion $i_{-1}(\ker(u_2)) \subseteq \ker \overline{(j_{-1}, u)}$. The isomorphism between the quotients $\ker(j_{-1})/\mathrm{im}(i_{-1})$ and $\ker(j_0)/\mathrm{im}(i_0)$ induces the exact sequence

$$0 \longrightarrow \mathrm{im}(i_{-1}) \longrightarrow \ker(j_{-1}) \stackrel{u}{\longrightarrow} \ker(j_0)/\mathrm{im}(i_0) \longrightarrow 0$$

Therefore we have the equality $\ker \overline{(j_{-1}, u)} = \mathrm{im}(i_{-1})$. Now because of the commutativity of the first square of diagram (3.1) and because of the injectivity of i_0 we have that $i_{-1}(\ker(u_2))$ contains $\ker \overline{(j_{-1}, u)}$. Hence we can conclude that via the morphism i_{-1}, $\ker(u_2)$ and $\ker \overline{(j_{-1}, u)}$ are isomorphic.
In order to prove that $\pi_0(st(M_2))$ is the kernel of the surjection $\pi_0(J) : \pi_0(st(M)) \to \pi_0(st(M_1))$ consider the following commutative diagram

(3.4)
$$\begin{tikzcd}
G_2 \arrow[r, "i_0"] \arrow[d] & G \arrow[r, "j_0"] \arrow[d] & G_1 \arrow[d] \\
G_2/u_2(X_2) \arrow[r, "\pi_0(I)"] & G/u(X) \arrow[r, "\pi_0(J)"] & G_1/u_1(X_1)
\end{tikzcd}$$

Since $J \circ I = 0$, it is clear that $\pi_0(I)(\pi_0(st(M_2)))$ is contained in the kernel of the surjection $\pi_0(J) : \pi_0(st(M)) \to \pi_0(st(M_1))$. Moreover there is a surjection between the kernel of j_0 and the kernel of $\pi_0(J)$. For $\ker(j_0)$ there are two possibilities: either it is contained in $\mathrm{im}(i_0)$ or it is contained in $\ker(j_0)/\mathrm{im}(i_0)$. Because of the commutativity of the first square of (3.4), the first case implies that $\ker(\pi_0(J))$ is contained in $\pi_0(I)(\pi_0(st(M_2)))$. The second case gives no informations about $\ker(\pi_0(J))$ since the quotients $\ker(j_0)/\mathrm{im}(i_0)$ and $\ker(j_{-1})/\mathrm{im}(i_{-1})$ are isomorphic via the morphism $u : X \to G$. □

4. Proof of the conjecture

First we recall briefly the construction of the Hodge, De Rham and ℓ-adic realizations of a 1-motive $M = (X, A, T, G, u)$ defined over S (see [D1] §10.1 for more details):

- if S is the spectrum of the field $\mathbb{C}$ of complex numbers, the Hodge realization $T_{\mathrm{H}}(M) = (T_{\mathbb{Z}}(M), \mathrm{W}_*, \mathrm{F}^*)$ of M is the mixed Hodge structure consisting of the fibred product $T_{\mathbb{Z}}(M) = \mathrm{Lie}(G) \times_G X$ (viewing $\mathrm{Lie}(G)$ over G via the exponential map and X over G via u) and of the weight and Hodge filtrations defined in the following way:
$$\begin{aligned}
\mathrm{W}_0(T_{\mathbb{Z}}(M)) &= T_{\mathbb{Z}}(M),\\
\mathrm{W}_{-1}(T_{\mathbb{Z}}(M)) &= \mathrm{H}(G,\mathbb{Z}),\\
\mathrm{W}_{-2}(T_{\mathbb{Z}}(M)) &= \mathrm{H}(T,\mathbb{Z}),\\
\mathrm{F}^0(T_{\mathbb{Z}}(M)\otimes\mathbb{C}) &= \ker(T_{\mathbb{Z}}(M)\otimes\mathbb{C} \longrightarrow \mathrm{Lie}(G)).
\end{aligned}$$
- if S is the spectrum of a field k of characteristic 0 embeddable in $\mathbb{C}$, the ℓ-adic realization $\mathrm{T}_\ell(M)$ of the 1-motive M is the projective limit of the $\mathbb{Z}/\ell^n\mathbb{Z}$-modules $\mathrm{T}_{\mathbb{Z}/\ell^n\mathbb{Z}}(M)$,
$$\begin{aligned}
\mathrm{T}_{\mathbb{Z}/\ell^n\mathbb{Z}}(M) &= \mathrm{H}^0(M \overset{\mathbb{L}}{\otimes} \mathbb{Z}/\ell^n\mathbb{Z})\\
&= \Big\{(x,g)\in X\times G \mid u(x)=\ell^n\, g\Big\}\Big/\Big\{(\ell^n\, x, u(x)) \mid x\in X\Big\},
\end{aligned}$$
where M is considered as a complex concentrated in degree 0 and 1 and $[\mathbb{Z}\xrightarrow{\ell^n}\mathbb{Z}]$ is a complex concentrated in degree -1 and 0.
- if S is the spectrum of a field k of characteristic 0 embeddable in $\mathbb{C}$, the De Rham realization $\mathrm{T}_{\mathrm{dR}}(M)$ of M is the Lie algebra of $G^\natural$ where $M^\natural = [X \to G^\natural]$ is the universal vectorial extension of M. The Hodge filtration on $\mathrm{T}_{\mathrm{dR}}(M)$ is defined by $\mathrm{F}^0\mathrm{T}_{\mathrm{dR}}(M) = \ker(\mathrm{Lie}\, G^\natural \to \mathrm{Lie}\, G)$.

Let S be the spectrum of a field k of characteristic 0 embeddable in $\mathbb{C}$. Fix an algebraic closure $\overline{k}$ of k. Let $\mathcal{MR}_{\mathbb{Z}}(k)$ be the integral version of the neutral Tannakian category over $\mathbb{Q}$ of mixed realizations (for absolute Hodge cycles) over k. The objects of $\mathcal{MR}_{\mathbb{Z}}(k)$ are families
$$N = ((N_\sigma, \mathcal{L}_\sigma), N_{\mathrm{dR}}, N_\ell, I_{\sigma,\mathrm{dR}}, I_{\overline{\sigma},\ell})_{\ell,\sigma,\overline{\sigma}}$$
where

- N_σ is a mixed Hodge structure for any embedding $\sigma : k \to \mathbb{C}$ of k in $\mathbb{C}$;
- N_{dR} is a finite dimensional k-vector space with an increasing filtration W_* (the Weight filtration) and a decreasing filtration F^* (the Hodge filtration);
- N_ℓ is a finite-dimensional $\mathbb{Q}_\ell$-vector space with a continuous $\mathrm{Gal}(\overline{k}/k)$-action and an increasing filtration W_* (the Weight filtration), which is $\mathrm{Gal}(\overline{k}/k)$-equivariant, for any prime number ℓ;
- $I_{\sigma,\mathrm{dR}} : N_\sigma \otimes_{\mathbb{Q}} \mathbb{C} \to N_{\mathrm{dR}} \otimes_k \mathbb{C}$ and $I_{\overline{\sigma},\ell} : N_\sigma \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \to N_\ell$ are comparison isomorphisms for any ℓ, any σ and any $\overline{\sigma}$ extension of σ to the algebraic closure of k;
- $\mathcal{L}_\sigma$ is a lattice in N_σ such that, for any prime number ℓ, the image $\mathcal{L}_\sigma \otimes \mathbb{Z}_\ell$ of this lattice through the comparison isomorphism $I_{\overline{\sigma},\ell}$ is a $\mathrm{Gal}(\overline{k}/k)$-invariant subgroup of N_ℓ ($\mathcal{L}_\sigma$ is the integral structure of the object N of $\mathcal{MR}_{\mathbb{Z}}(k)$).

Before to define the morphisms of the category $\mathcal{MR}_{\mathbb{Z}}(k)$ we have to introduce the notion of Hodge cycles and of absolute Hodge cycles. Let $N = ((N_\sigma, \mathcal{L}_\sigma), N_{\mathrm{dR}}, N_\ell, I_{\sigma,\mathrm{dR}}, I_{\overline{\sigma},\ell})_{\ell,\sigma,\overline{\sigma}}$ be an object of the Tannakian category $\mathcal{MR}_{\mathbb{Z}}(k)$. A Hodge cycle of N relative to an embedding $\sigma : k \to \mathbb{C}$ is an element $(x_\sigma, x_{\mathrm{dR}}, x_\ell)_\ell$ of $N_\sigma \times N_{\mathrm{dR}} \times \prod_\ell N_\ell$ such that $I_{\sigma,\mathrm{dR}}(x_\sigma) = x_{\mathrm{dR}}, I_{\overline{\sigma},\ell}(x_\sigma) = x_\ell$ for any prime number ℓ and

$x_{\mathrm{dR}} \in \mathrm{F}^0 N_{\mathrm{dR}} \bigcap \mathrm{W}_0 N_{\mathrm{dR}}$. An absolute Hodge cycle is a Hodge cycle relative to every embedding $\sigma : k \to \mathbb{C}$. By definition, the morphisms of the Tannakian category $\mathcal{MR}_{\mathbb{Z}}(k)$ are the absolute Hodge cycles: more precisely, if N and N' are two objects of $\mathcal{MR}_{\mathbb{Z}}(k)$, the morphisms $\mathrm{Hom}_{\mathcal{MR}_{\mathbb{Z}}(k)}(H, H')$ are the absolute Hodge cycles of the object $\underline{\mathrm{Hom}}(H, H')$.

Since 1-motives are endowed with an integral structure, according to [D1] (10.1.3) we have the fully faithful functor

$$\begin{array}{ccc} \{1-\mathrm{Mot}(k)\} & \longrightarrow & \mathcal{MR}_{\mathbb{Z}}(k) \\ M & \longmapsto & \mathrm{T}(M) = (\mathrm{T}_\sigma(M), \mathrm{T}_{\mathrm{dR}}(M), \mathrm{T}_\ell(M), I_{\sigma,\mathrm{dR}}, I_{\overline{\sigma},\ell})_{\ell,\sigma,\overline{\sigma}} \end{array}$$

which attaches to each 1-motive M of $\mathcal{M}(k)$ its Hodge realization $\mathrm{T}_\sigma(M) = (\mathrm{T}_\sigma(M), \mathcal{L}_\sigma)$ with integral structure for any embedding $\sigma : k \to \mathbb{C}$ of k in $\mathbb{C}$, its de Rham realization $\mathrm{T}_{\mathrm{dR}}(M)$, its ℓ-adic realization $\mathrm{T}_\ell(M)$ for any prime number ℓ, and its comparison isomorphisms.

We can now prove the conjecture of Deligne cited in the Introduction:

Proof. Let M_1 and M_2 be two 1-motives defined over k. Denote by $\mathrm{T}(M_1) = (\mathrm{T}_\sigma(M_1), \mathrm{T}_{\mathrm{dR}}(M_1), \mathrm{T}_\ell(M_1), I_{\sigma,\mathrm{dR}}, I_{\overline{\sigma},\ell})$ and $\mathrm{T}(M_2) = (\mathrm{T}_\sigma(M_2), \mathrm{T}_{\mathrm{dR}}(M_2), \mathrm{T}_\ell(M_2), I_{\sigma,\mathrm{dR}}, I_{\overline{\sigma},\ell})$ the system of realizations defined by M_1 and M_2 respectively. Consider an extension of $\mathrm{T}(M_1)$ by $\mathrm{T}(M_2)$ in the category $\mathcal{MR}_{\mathbb{Z}}(k)$:

$$0 \longrightarrow \mathrm{T}(M_2) \longrightarrow N \longrightarrow \mathrm{T}(M_1) \longrightarrow 0$$

with $N = (N_\sigma = (N_\sigma, \mathcal{L}_\sigma), N_{\mathrm{dR}}, N_\ell, I_{\sigma,\mathrm{dR}}, I_{\overline{\sigma},\ell})$. Such an extension implies an extension for each realization:

$$0 \longrightarrow \mathrm{T}_*(M_2) \longrightarrow N_* \longrightarrow \mathrm{T}_*(M_1) \longrightarrow 0$$

with $* = \sigma, \mathrm{dR}$ oder ℓ.
According to Definition (3.1) an extension of M_1 by M_2 is a 1-motive M and two morphisms of 1-motives $(i_{-1}, i_0) : M_2 \to M$ and $(j_{-1}, j_0) : M \to M_1$ such that $j_{-1} \circ i_{-1} = 0, j_0 \circ i_0 = 0$, i_{-1} and i_0 are injective, j_{-1} and j_0 are surjective, and u induces an isomorphism between the quotients $\ker(j_{-1})/\mathrm{im}(i_{-1})$ and $\ker(j_0)/\mathrm{im}(i_0)$. By Proposition 3.2, via the equivalence of categories (1.8), the strictly commutative Picard **S**-stack $st(M)$ is an extension of the strictly commutative Picard **S**-stack $st(M_1)$ by the strictly commutative Picard **S**-stack $st(M_2)$:

$$st(M_2) \xrightarrow{I} st(M) \xrightarrow{J} st(M_1)$$

where the morphisms of strictly commutative Picard **S**-stacks I and J are induced by the morphisms of 1-motives (i_{-1}, i_0) and (j_{-1}, j_0). In particular by Lemma 1.8, M_2 is quasi-isomorphic to the kernel of J. For each realization $* = \sigma, \mathrm{dR}$ oder ℓ, we have therefore an exact sequence

$$0 \longrightarrow \mathrm{T}_*(M_2) \stackrel{\mathrm{T}_*(i_{-1},i_0)}{\longrightarrow} \mathrm{T}_*(M) \stackrel{\mathrm{T}_*(j_{-1},j_0)}{\longrightarrow} \mathrm{T}_*(M_1) \longrightarrow 0$$

where the morphisms $\mathrm{T}_*(i_{-1}, i_0)$ and $\mathrm{T}_*(j_{-1}, j_0)$ are induced by the morphisms of 1-motives (i_{-1}, i_0) and (j_{-1}, j_0). Choosing adequately the extension M, for each realization $* = \sigma, \mathrm{dR}$ oder ℓ we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{T}_*(M_2) & \longrightarrow & \mathrm{T}_*(M) & \longrightarrow & \mathrm{T}_*(M_1) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathrm{T}_*(M_2) & \longrightarrow & N_* & \longrightarrow & \mathrm{T}_*(M_2) & \longrightarrow & 0. \end{array} \tag{4.1}$$

Therefore we can conclude that the system of realizations $N = (N_\sigma = (N_\sigma, \mathcal{L}_\sigma), N_{\mathrm{dR}}, N_\ell, I_{\sigma,\mathrm{dR}}, I_{\overline{\sigma},\ell})$ is isomorphic to the system of realizations defined by the 1-motive M:

$$N \cong \mathrm{T}_*(M).$$

□

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